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ON THE PICARD-LEFSCHETZ TRANSFORMATION FOR ALGEBRAIC MANIFOLDS ACQUIRING GENERAL SINGULARITIES

BY

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ABSTRACT. We consider a holomorphic family $\{V_t\}_{t\in D}$ of projective algebraic varieties V_t parametrized by the unit disc $D = \{t \in \mathbf{C}: |t| < 1\}$ and where V_t is smooth for $t \neq 0$ but V_0 may have arbitrary singularities. Displacement of cycles around a path $t = t_0e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ leads to the Picard-Lefschetz transformation T: $H_*(V_{t0}, \mathbf{Z}) \to H_*(V_{t0}, \mathbf{Z})$ on the homology of a smooth V_{t0}. We prove that the eigenvalues of T are roots of unity and obtain an estimate on the elementary divisors of T. Moreover, we give a global inductive procedure for calculating T in specific examples, several of which are worked out to illustrate the method.

1. Introduction.

A. Let W and W' be algebraic varieties and f: $W \to W'$ a rational map. For $t \in W'$, let $V_t = f^{-1}(t)$ be the corresponding fibre.

In general, there exists an algebraic subset S of W' such that $W - f^{-1}(S)$ is a fibre space over $W' - S$ via f. This induces a representation of the fundamental group $\pi_1(W' - S, t)$ as a group of homeomorphisms of V_t. Hence, $\pi_1(W' - S, t)$ acts as a group of linear transformations on the homology $H_*(V_t)$.

This leads to the following problem:

Given a loop in $W' - S$, describe the corresponding transformation.

B. As it stands, this problem is too general. We will make the following simplifying assumptions:

(1.1) W is projective and nonsingular.

(1.2) W' is a projective, nonsingular curve C. This means that the fibres V_t are divisors on W, i.e., their components may have multiplicities greater than 1.

Assumption (1.2) is useful because in many cases $\pi_1(W' - S, t)$ is generated by $\pi_1(C - C \cap S, t)$, where C is a general curve in W' containing t, e.g., a plane section relative to some projective embedding of W'. When W' is a projective space, this was shown by Zariski [8].

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(1.3) f is defined on all of W, i.e., there is no base locus. (This is not essential and will be weakened later on.)

With these assumptions, Bertini's theorem tells us that all the V_t are nonsingular except for those t belonging to a certain finite subset of C. These exceptional points form the set S mentioned above.

(1.4) The loop g in which we are interested is a simple closed curve containing just one point p of S.

The action on homology induced by g depends only on the nature of the singularities of V_p. When V_p has only ordinary double points, Picard [6] and Lefschetz [4] gave a formula for expressing the transformation. This is why we call the action in more general situations the Picard-Lefschetz transformation (or just the P-L transformation).

(1.5) V_p has only normal crossings, i.e., its underlying algebraic set is the union of nonsingular hypersurfaces of W meeting transversally (cf. part 9B).

C. We begin by describing Lefschetz's method for obtaining the homology of an algebraic variety by means of hyperplane sections. This is applied to each fibre V_t, for t near p, to yield the following results:

(1.6) **Theorem I.** *The eigenvalues of the P-L transformation are roots of unity.*

More precisely, we have

(1.7) **Theorem I′.** *Let $m_1, \cdots, m_r$ be the multiplicities of the components of V_p. Then each eigenvalue of the P-L transformation is an m_ith root of unity for some $i = 1, \cdots, r$.*

(1.8) **Theorem II.** *The elementary divisors of the P-L transformation are all of degree $\leq \dim W$.*

More precisely, we have

(1.9) **Theorem II′.** *Let s be the maximum number of components of V_p which have a point in common. Then the elementary divisors of the P-L transformation are all of degree $\leq s$.*

We conclude by examining the P-L transformation in some specific cases.

2. **The 0-dimensional case.**

A. When W itself is a curve, so that the V_t are 0-dimensional, the P-L transformation can be described through the classical theory of algebraic functions of one variable.

Let $x \in W$ and let z be a local coordinate at x in W. Let t be a local coordinate at $f(x)$ in C, so that near x the function f is given by

$$t = f(z) = c_1 z^m + c_2 z^{m+1} + \cdots \tag{2.1}$$

where $c_1 \neq 0$ and m is the multiplicity of x as a component of $V_0 = f^{-1}(f(x))$.

The power series (2.1) can be inverted to yield the Puiseux series

$$z = (1/c_1)t^{1/m} + \cdots \tag{2.2}$$

which is a power series in $t^{1/m}$. Hence, when t is near 0, V_t consists of m points near x and these points permute cyclically as t makes one circuit about 0.

Note that the fibre V_0 consists of normal crossings and the P-L transformation satisfies the theorems stated in part 1C.

B. The homology of the curve W can be determined in a classical way by means of the pencil V_t.

For simplicity, let us assume that the parameter curve C is actually the projective line P^1. There will be a finite set R in W at which f ramifies, i.e., points where the m of (2.1) is > 1. Let $S = f(R)$. Then $t \in S$ if and only if V_t is singular, i.e., $V_t = m_1 x_1 + \cdots + m_r x_r$, where the x_i are distinct points of W, the m_i are positive integers and at least one m_i is > 1. The behavior of the points of W near the x_i was mentioned above.

From the continuity of algebraic functions, we deduce the following:

(2.3) $f\colon W - f^{-1}(S) \to P^1 - S$ is a fibre space with fibre any nonsingular V_t.

(2.4) If $g\colon [0, 1] \to P^1$ is a continuous map of the unit interval into P^1 such that $g[0, 1) \subseteq P^1 - S$ and if $y \in f^{-1}(g(0))$, then there is a unique map $\bar{g}$: $[0, 1] \to W$ such that $f \circ \bar{g} = g$ and $\bar{g}(0) = y$.

(2.4) implies that there is a continuous map T_g from $f^{-1}(g(0))$ to $f^{-1}(g(1))$. In particular, $\pi_1(P^1 - S, p)$ acts on $H_0(f^{-1}(p))$.

(2.5) In case $g(1) \in S$ and $f^{-1}(g(1)) = m_1 x_1 + \cdots + m_r x_r$, then T_g is onto and $T_g^{-1}(x_i)$ consists of m_i points.

This leads us to two important concepts. The first is that of *vanishing cycle* (along g), which is just an element d of $H_0(f^{-1}(g(0)))$ such that $T_g(d) = 0$ in $H_0(f^{-1}(g(1)))$. In the case at hand, the vanishing cycles are generated by those of the form $y_1 - y_2$, where $T_g(y_1) = T_g(y_2)$.

Next, we have the concept of the *cylinder* generated by a cycle d in $H_0(f^{-1}(g(0)))$. We simply keep track of the cycle in each fibre over $g[0, 1]$. This cylinder, denoted $D(d, g)$, is a 1-chain which, with appropriate orientation, has boundary $d - T_g(d)$.

In the special case where d vanishes along g, $\partial D(d, g) = d$ and we can speak of the *cone* generated by d. It is an element of the relative group $H_1(W, f^{-1}(g(0)))$.

(2.6) Pick a base-point p in P^1 and let $\{g_i\}$ be simple paths from p to the points of S meeting only at p. Then $H_1(W, V_p)$ is generated by the cones over the g_i.

In particular, if we assume that the singular fibres are as simple as possible, viz., of the form $2x_1 + x_2 + \cdots + x_r$ (the x_i being distinct points of W), then there is exactly one vanishing cycle d_i for each g_i. d_i will be of the form $u_i - v_i$, where u_i, v_i are points in V_p. Hence,

(2.7) $(d_i, d_i) = 2$, where $(\ ,\ )$ is the Kronecker intersection pairing in $H_0(V_p)$.

If we replace g_i by the corresponding loop around its endpoint in S, this loop induces the identity transformation on V_p except that u_i and v_i are interchanged. This can be expressed as

(2.8) The P-L transformation is given by the formula $b \to b - (b, d_i)d_i$, for $b \in H_0(V_p)$.

3. The Lefschetz theorems.

A. The P-L transformation we seek can be considered a local problem. That is, we have a pencil V_t on W and we wish to examine its (homology) behavior near one (singular) member. This was described in part 2A when W is a curve.

But there is also a global theory analogous to (2.6) in which the pencil determines the homology of the ambient W. This was developed by Lefschetz [4] and is described below. Using his global construction of homology in a given dimension (that of the V_t), we can determine the (local) P-L transformation in the dimension one higher.

B. At this point we mention some topological facts which generalize (2.3) and (2.4).

We have our pencil of divisors V_t given as the fibres under the rational map f: $W \to C$, where C is a curve. If $\dim W > 1$, the first new phenomenon we meet is that f may not be defined on all of W, i.e., there may be a base locus B of codimension 2 in W which is common to all the V_t. For the purposes of the present section, this will be more of a help than a hindrance (see part 11 for more on pencils with base).

Each fibre V_t has a tubular neighborhood in W, i.e., a neighborhood of which it is a deformation retract, and if t' is near enough to t, then $V_{t'}$ will be in that neighborhood. Hence, there is an associated map $g_{t't}$: $V_{t'} \to V_t$ unique up to homotopy. If V_t is nonsingular, then $V_{t'}$ will be a cross-section of the tubular neighborhood and $g_{t't}$ will be a homeomorphism.

We assume that the general V_t is nonsingular and irreducible. Then there is a finite subset $S \subseteq C$ such that V_t is singular if and only if $t \in S$. Using the tubular neighborhoods, $W - f^{-1}(S) - B$ becomes a fibre bundle over $C - S$ via f (see Wallace [7]).

Hence, if g: $[0, 1] \to C$ is a path such that $g[0, 1) \subseteq C - S$, then $V_{g(1)}$ is a deformation retract of $\bigcup_{v \in [0,1]} V_{g(v)}$ and if d is a subset or cycle in $V_{g(0)}$, we will have a corresponding cylinder unique up to homotopy. The cylinders over homotopic paths are homologous. As in part 2B, we have the notions of vanishing cycle, cone, etc.

C. Lefschetz first finds the P-L transformation when the singular fibre V_0 has an ordinary double point off the base.

An ordinary double point can be described in several equivalent ways:

x is an ordinary double point for V_0 if

(3.1) the tangent cone to V_0 at x is of degree 2 without multiple generators, or

(3.2) if $z_1, \cdots, z_n$ are local coordinates at x in W, then V_0 is given locally by an equation $H(z_1, \cdots, z_n) = 0$, where

(1) $\partial H/\partial z_i|_x = 0$ for all i, that is, x is singular for V_0,

(2) $|\partial^2 H/\partial z_i \partial z_j|_x \neq 0$, that is, the partials of H form a local coordinate system at x,

or,

(3.3) local coordinates $z_1, \cdots, z_n$ can be found so that V_0 is given by $z_1^2 + \cdots + z_n^2 = 0$ (Fáry [1]).

Hence, if t is a local coordinate at $f(x)$ in C, then the pencil can be given locally by $z_1^2 + \cdots + z_n^2 = t$. We then have

(3.4) The P-L transformation from $H_i(V_t)$ to itself is the identity if $i \neq n-1$.

(3.5) If $d_1 = \{z_i \text{ real } | z_1^2 + \cdots + z_n^2 = 1\}$ and $d_t = t^{-1/2} d_1$, then d_t is an $(n-1)$-sphere in V_t $(t \neq 0)$ representing a homology class in $H_{n-1}(V_t)$ such that $d_t \to 0$ as $t \to 0$, i.e., d_t is a vanishing cycle on V_t.

(3.6) If $c \in H_{n-1}(V_t)$, then c goes to $c + (-1)^{n(n+1)/2}(c, d_t)d_t$ as t goes around 0.

(3.7) If n is odd, d_t goes to $-d_t$ and $(d_t, d_t) = (-1)^{(n-1)/2}2$; if n is even, d_t goes to d_t and $(d_t, d_t) = 0$.

D. Lefschetz's global theorems require that the pencil V_t belong to a "sufficiently general" linear system V. According to Lefschetz, this means

(3.8) $\dim |V| \geq n$ and any n members of $|V|$ have a point in common.

These properties, of course, are satisfied by any pencil on a curve, which fits in with the results of part 2. They are also satisfied by the system of hyperplane sections of W (relative to some projective embedding), and this is the system we will be using (see part 4).

As we did before, we cut the parameter line by picking a base-point $p \notin S$ and tracing nonintersecting cuts g_i from p to the points of S. Then we have

(3.9) $H_i(W, V_t) = 0$, $i \leq n - 1 = \dim V_t$ (for any V_t).

(3.10) $H_n(W, V_p)$ is generated by the cones over the g_i.

These hold regardless of the nature of the singular fibres. It is often convenient to assume that the singular V_t have one ordinary double point off the base (see part 4A for such a possibility), so that (3.6) tells us that there is just one cone for each g_i.

Let $D_i = D(d_i, g_i)$. What are the relations among the D_i in $H_n(W, V_p)$? There is one obvious way to generate such relations:

Let $U = P^1 - \bigcup_i g_i$. Then U is homeomorphic to a disc and f: $W - f^{-1}(\bigcup_i g_i) \to U$ is the trivial bundle. If $V_t - B$ is a typical fibre $(t \in U)$, pick a (compact or noncompact) cycle c in it. (This is really the same as a compact cycle in $H_{n-1}(V_t, B)$.)

Now $c \times U$ is a noncompact $(n+1)$-cycle in $W - f^{-1}(\bigcup_i g_i)$. It can be thought of as the "translates" of c to the other fibres over U (these are coherent, because the bundle is trivial). What is $\partial(c \times U)$ in W? It is the n-chain $(\partial c) \times U + c \times (\partial U)$ (viewing c now in $H_{n-1}(V_t, B)$).

$(\partial c) \times U$ are the translates of ∂c in the fibres V_t $(t \in U)$, but since $\partial c \subseteq B$ and B is always left point-wise fixed under the various deformation retracts (part 3B), the collection of translates of ∂c consists of ∂c itself. But this is only $(n-1)$-dimensional and so does not figure effectively in $\partial(c \times U)$.

As for $c \times (\partial U)$ (or, at least, its n-dimensional pieces), observe that each g_i contributes two edges to ∂U in the sense that as c approaches the same point t of g_i (except p) from opposite sides, the resulting cycles may not be homologous in $H_{n-1}(V_t)$. In fact, by (3.6), they differ by a multiple of d_i (i.e., after d_i has been put in V_t via g_i). Hence, as t traces g_i, c traces a certain multiple of D_i.

Hence, for some integers n_i, $\partial(c \times U) = n_1 D_1 + \cdots +$ something in V_p (p being the only other part of ∂U), i.e., we have a relation in $H_n(W, V_p)$. Then,

(3.11) All relations among the D_i arise this way.

(3.12) If $c \in H_{n-1}(V_t, B)$ yields the trivial relation, then $\partial c = 0$ and c is invariant under all the P-L transformations, i.e., $(c, d_i) = 0$ for all i.

Hence,

(3.13) rank $H_n(W, V_p) = m -$ rank $H_{n-1}(V_t, B) +$ rank $H_{n-1}(W)$, where m is the number of points in S and V_t is any nonsingular fibre.

E. Suppose now the V_t are hyperplane sections and let H be one of them (one can think of it as the "hyperplane at infinity"). Then $W - H$ and the $V_t - H$ are affine varieties and we have (Fáry [1], Žižčenko [9])

(3.14) $H_i(W - H) = 0$, $i > \dim W$ and similarly,

(3.15) $H_i(V_t - H) = 0$, $i > \dim V_t$ for any t.

(3.16) $H_i(W - H, V_t - H) = 0$, $i < n = \dim W$.

(3.17) $H_n(W - H, V_t - H)$ is generated by the D_i and these are independent, i.e., $H_n(W - H, V_t - H) \simeq \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (m times).

4. **Hyperplane sections.**

A. Parts 3D and 3E show how important hyperplane sections are for describing the homology of varieties. We now mention some facts about hyperplane sections that will be useful later on (see Wallace [7]).

If V is a nonsingular irreducible variety in a projective space P_N, then the general hyperplane section of V is nonsingular (Bertini's theorem). Hence, the collection of hyperplanes cutting singular sections on V (or containing V entirely) forms a proper irreducible subvariety V^* of the dual space P_N^*. V^* is called the *dual* of V. We assume that V is not just a linear subspace of P_N so that V^* is a hypersurface in P_N^*. Its degree m is called the *class* of V.

A nonsingular point H of V^* corresponds to a section of V with one ordinary double point (part 3C) and the tangent plane to V^* at H consists of those hyperplanes of P_N which pass through that double point.

Now take a line L in P_N^* which is in general position with respect to V^*. L just represents a pencil of hyperplanes in P_N. Since L is general, it meets V^* in m distinct points. These points are nonsingular for V^* and L is not tangent to V^* at any of them. Hence, L is the kind of pencil we have considered before (part 3D), viz., one in which each singular fibre has one ordinary double point off the base.

We will refer to such a pencil simply as *general* (with respect to V).

B. Now if we associate to a point in V the point in L on whose section it lies, we get a rational map $f\colon V \to L$ whose fibres are the hyperplane sections. We can pick a base-point in L, make cuts in L, look at the corresponding cones in V, etc.

More generally, we can do what was mentioned in 3B: let $H \in P_N^*$ (and suppose $V \not\subseteq H$). Then $V \cap H$ is locally a deformation retract in V and if U is any pre-assigned neighborhood of $V \cap H$, then $V \cap H$ will be $\subseteq U$ as soon as H' is near enough to H in P_N^*.

This means that if $g\colon [0, 1] \to P_N^*$ is a family of hyperplanes and if $g(u) \cap V$ has the same kind of singularities as $g(u') \cap V$ for $u, u' \in [0, 1)$, then there are induced homeomorphisms from $g(u) \cap V$ to $g(u') \cap V$ and an induced map from $g(u) \cap V$ to $g(1) \cap V$ $(u \in [0, 1])$. We get the usual collection of vanishing cycles, cones, etc.

For example, if $g(u)$ is a nonsingular point in V^* for all $u \in [0, 1]$, then the function h from $[0, 1]$ to V which assigns to each u the double point of $g(u) \cap V$ is continuous, and the rest of the homeomorphism between $g(u) \cap V$ and $g(u') \cap V$ can be obtained by construction of the normal bundles in $V - h[0, 1]$.

C. Suppose now that V is no longer nonsingular and irreducible. What should V^* mean? It cannot just mean those hyperplanes whose section on V is singular, since if the dimension of the singular set of V is > 0, every hyperplane does that.

What V^* ought to be is those sections which are "more singular" than the general section. I am not sure what that would be in general. Perhaps one could show that for all but a proper algebraic subset of P_N^* the sections are all homeomorphic, or (if the Betti numbers are semicontinuous functions on P_N^*) have the same Euler characteristic or middle homology group.

For instance, if V has isolated singularities (or none), these criteria are all equivalent and V^* is the planes tangent to V at nonsingular points or passing through a singular point (i.e., cutting singular sections).

D. Furthermore, if one knew what V^* was, one could then take a general pencil L in which a finite number of sections would be "extra-singular." Again there would be vanishing cycles and cones with which one could hope to build the homology of V.

(3.9) would fail, but perhaps (3.10) could become

(4.1) $H_i(V, H \cap V)$ is generated by the cones for $i \leq \dim V$.

For example, suppose V possesses an improper double point, i.e., one formed by taking a nonsingular $\overline{V}$ and identifying two of its points, P and Q. An arc in $\overline{V}$ joining P and Q becomes a 1-cycle in V which cannot be pulled off the double point. So if $\dim V > 1$, (3.9) is violated, but clearly this 1-cycle is homologous to a cone.

These ideas can be worked out for curves, at least, and also for certain simple kinds of singular varieties (see part 9).

5. The branch function.

A. We return to our pencil V_t given by $f\colon W \to C$. For the present we allow a base locus but assume the general V_t nonsingular (unless we can apply 4C and D). We have the finite set S whose fibres are singular. Then from 4A, V_t^* $(t \notin S)$ is a hypersurface in P_N^* and forms an algebraic family (indexed by $C - S$). Hence, all the V_t $(t \notin S)$ have the same class m, and when t approaches a point 0 in S, V_t^* becomes a well-defined (reducible) hypersurface in P_N^* of degree m.

This hypersurface, of course, will be defined to be the dual of V_0 and denoted V_0^*. It is another candidate to be added to those suggested in 4C. However, it is not intrinsic, since it depends on the enveloping pencil V_t, at least if there is a base locus B present. In general, $V_0^* =$ dual of V_0 (viz. 4C) + dual of singular locus of B (viz. 4C). See part 9 below for a special case.

B. The duals V_t^* now form a 1-parameter family of hypersurfaces of degree m indexed by all of C. Let L be a line in P_N^* in general position with respect to this family.

(5.1) **Theorem.** *Every* V_t^* *meets* L *in a finite number of points (which, of course, is always* $\leq m$*).*

(5.2) **Proof.** The Grassmann variety $G(1, N)$ of lines in P_N is of dimension $2N - 2$. Those lines contained in some V_t^* form an algebraic subset $\mathfrak{L}$ of $G(1, N)$. We need only show that $\dim \mathfrak{L} \leq 2N - 3$.

The principle of counting constants shows that we need only establish

(5.3) **Lemma.** *Let* V *be a hypersurface in* P_N. *Then the lines in* V *form* $\mathfrak{L}$ *an algebraic subset* $\mathfrak{L}$ *of* $G(1, N)$ *of dimension* $\leq 2N - 4$.

(5.4) **Proof.** Let H be a hyperplane and p a point not in V or H. Then if L is any line in V, p and L span a 2-plane which meets H in a line. We thus establish a rational map h from $\mathfrak{L}$ to $G(1, N - 1)$ which is everywhere defined. Since $\dim G(1, N - 1) = 2N - 4$, we need only show that $h^{-1}(L)$ is finite, where L is a line in H.

If not, then $h^{-1}(L)$ would contain a 1-parameter family of lines, a ruled subsurface R of V which p projects onto the line L. Then R is a subset of the 2-plane (p, L) spanned by p and R, hence, $R = (p, L)$. But a (nontrivial) 1-parameter family of lines in a plane must cover it, i.e., one of the lines of $h^{-1}(L)$ passes through p. But this contradicts the assumption that $p \notin V$.

C. (5.1) shows that $\{V_t^* \cap L\}$, $t \in C$, is a γ_m^1 on L, i.e., a 1-parameter family of divisors of degree m, if L is general. It clearly has no base points, since these would be common to all the V_t^*. But $\bigcap_{t \in C} V_t^*$ is of codimension ≥ 2 in P_N^*, so a general line misses it.

All but a finite number of the m-tuples in γ_m^1 consist of m distinct points, i.e., L is general with respect to all but a finite number of the V_t^*.

Let $\overline{S}$ = those t for which L is not general. Clearly, $S \subseteq \overline{S}$. Let u be a coordinate on L, so we can consider u as a rational map from W to L whose fibres are the hyperplane sections. Then γ_m^1 defines an algebraic correspondence between C and L: $t \in C$ and $u \in L$ are associated if $u \in V_t^*$. In this way u becomes an algebraic function $u(t)$ on C and $\overline{S}$ is where $u(t)$ ramifies.

(5.5) **Definition.** $u(t)$ is called the *branch function* of the pencil V_t relative to L. The correspondence curve $J = \{(t, u) \in C \times L \mid u \in V_t^*\}$ is called the *branch curve*.

D.

(5.6) **Theorem.** *The branch curve is irreducible.*

(5.7) **Proof.** We extend the correspondence J as follows:

Let M be a general 2-plane through L and consider the family $V_t^* \cap M$ in M. By (5.1), L is not contained in any component of the V_t, so M is not either.

Hence, each $V_t^* \cap M$ is a curve. Since a general V_t^* is irreducible and M is general, a general member of $\{V_t^* \cap M\}$ is also irreducible.

As before, we have a correspondence between C and M, viz., $\overline{J} = \{(t, v) \in C \times M \mid v \in V_t^*\}$. Its projection onto M is a finite covering, except that for finitely many $v \in M$, $C \times v \subseteq \overline{J}$. $\overline{J} \cap (t \times M) = V_t^* \cap M$, so $\overline{J}$ is an irreducible surface (note that no $t \times M$ is a component of $\overline{J}$).

Hence, we obtain a 2-dimensional linear system of curves on $\overline{J}$ (or on its desingularization) by pulling back the lines in M. But then Bertini's theorem says that the general member, i.e., J, is irreducible.

6. **Application of the branch function.**

A. We now describe how the branch function can be used to examine the homology behavior of the V_t.

Let $t_0 \notin \overline{S}$, so that L is general for V_{t_0}. Mark the m points $u(t_0)$ in L and pick a base-point u_0 in L distinct from the $u(t_0)$. As in 3D, draw cuts from u_0 to the $u(t_0)$. If we number the $u(t_0)$ as $u_1(t_0), \cdots, u_m(t_0)$, let the corresponding cuts be $g_1, \cdots, g_m$. Now as t moves in $C - \overline{S}$, starting from t_0, the $u_i(t)$ will vary continuously and never coincide, so we can make u_0 and the g_i vary continuously with t also, such that at any stage the $g_i(t)$ form an appropriate set of cuts in L centered at $u_0(t)$.

Let $d_i \in H_{n-2}(H_{u0} \cap V_{t0})$ be the vanishing cycle along $g_i(t_0)$. Then, as in 3B, as t moves in $C - \overline{S}$, the d_i will vary continuously so that $d_i(t) \in H_{n-2}(H_{u0(t)} \cap V_t)$ and $d_i(t)$ vanishes along $g_i(t)$. Then the corresponding cones $D_i(t)$ form a continuous family of $(n-1)$-chains, and if $\Sigma\, n_i D_i(t_0)$ is a cycle in $H_{u0} \cap V_{t0}$, then $\Sigma\, n_i D_i(t)$ forms a continuous family of $(n-1)$-cycles.

Suppose t traverses a loop g based at t_0. Then the points $u_i(t)$ describe m arcs in L, so there are points of L not on these arcs. This means that the base-point $u_0(t)$ can be chosen independent of t, i.e., is constantly equal to u_0 as t moves on g. Furthermore, we can choose u_0 so that $H_{u0} \cap W$ is nonsingular, since $L \not\subseteq W^*$. Hence, the $d_i(t)$ move in the pencil $H_{u0} \cap V_t$ and we have the basis for an inductive procedure.

The Lefschetz theorems of 3D show that any i-cycle of V_t $(i \leq n-2)$ can be put in $H_{u0} \cap V_t$, so the behavior of $H_i(V_t)$ can be reduced to a pencil of lower dimension. As for $H_i(V_t)$ $(i > n-1)$, the P-L transformation on it is adjoint to that on $H_{2n-2-i}(V_t)$ (at least mod torsion), so is determined by what happens on the first half of the homology sequence.

B. Thus, our problem becomes that of investigating the change in the cones over the g_i, i.e., the group $H_{n-1}(V_t, H_{u0} \cap V_t)$.

When t traverses g, the branch points $u_i(t_0)$ undergo a permutation π and the original cuts g_i are transformed into a new system of cuts $g'_{\pi(i)}$. The original vanishing cycles d_i and cones D_i become $d'_{\pi(i)}$ and $D'_{\pi(i)}$. We must express the D'_i in terms of the original D_i.

This is done as follows (cf. (3.11)):

Consider an arc g' in L from u_0 to $u_1(t_0)$, say. A certain $(n-2)$-cycle d vanishes along g'. Now in $L - \bigcup_{i>1} u_i(t_0)$, g' is homotopic to $h \circ g_1$ where h is a loop at u_0.

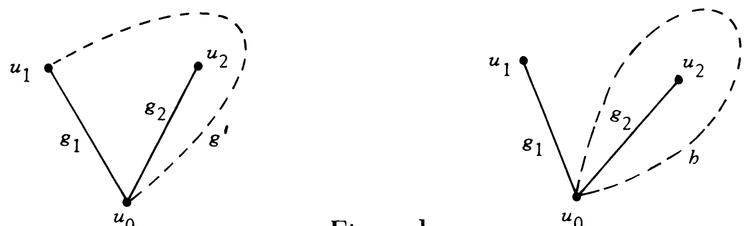

Figure 1

Now after t traverses h, d becomes a cycle $\bar{d}$ which vanishes along g_1. Hence, $\bar{d} = \pm d_1$. Choose d so that $\bar{d} = d_1$. Then $D(d, g') = D_1 + D(d, h)$. So we must reduce $D(d, h)$ to a sum of the D_i.

h, being an element of the fundamental group $\pi_1(L - u(t_0), u_0)$, is generated by loops l_i, where l_i encloses just g_i. Hence, $D(d, h) = \Sigma\, D(\bar{d}_i, l_i)$, but as in 3D, $D(\bar{d}_i, l_i) = \pm(\bar{d}_1, d_i)D_i$ and we are done.

C. Let us turn to the case at hand, where g is a small loop about a point 0 of S.

By the classical theory of algebraic functions of one variable, the branch function $u(t)$ can be expressed (near 0) by a collection of Puiseux series or cycles. We can choose the coordinate u on L so that H_∞ is not tangent to any V_t, t small. This means that the Puiseux series have no negative terms, i.e., have finite values at $t = 0$. These values, of course, are just the points $V_0^* \cap L$.

Consider one of these Puiseux series:

$$u(t) = c_1 t^{a_1/n} + c_2 t^{a_2/n} + \cdots, \tag{6.1}$$

where the c_i are nonzero constants and the a_i are integers with $a_1 < a_2 < \cdots$. Obviously, we can take the integers $n, a_1, a_2, \cdots$ relatively prime (so this cycle effectively contains n points of $V_t^* \cap L$). Furthermore, by choosing the u coordinate so that $u(0) = 0$, we can assume $a_1 > 0$.

These n points behave rather simply as t traverses g, viz., they undergo a cyclic permutation. But it is more difficult to say what happens to the cuts g_i. (See 7A for discussion and examples.) Here we will only discuss the case that we will need later (part 9).

D. The simplest behavior occurs when $a_1 = 1$. This just means that (0, 0) is a simple point on J. We can change the coordinate t on C so that (6.1) becomes $u = t^{1/n}$. The branch points then move in circles (if t does) and it is easy to choose cuts which behave nicely.

Suppose the points are numbered 1 through q and undergo the permutation $(12 \cdots q)$. For convenience, we picture them in a row (see Figure 2),

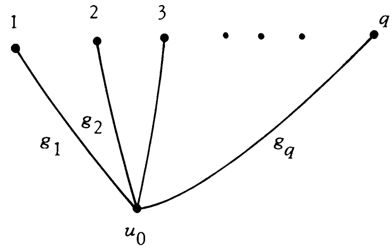

Figure 2

and draw the cuts g_i so that $g_1 \to g_2 \to \cdots \to g_q$ as t moves about 0. Furthermore, if d_1 vanishes along g_1 and $d_2, d_3, \cdots$ are its transforms in $H_{n-2}(V_t \cap H_{u_0})$ as t circles 0, then d_i vanishes along g_i, so the corresponding cones also undergo the transformation $D_1 \to D_2 \to \cdots \to D_q$.

The key question, then, is: what happens to D_q? The answer is $D_q \to D_{q+1} = D(d_{q+1}, g_{q+1})$, where d_{q+1} and g_{q+1} are the qth transforms of d_1 and g_1, respectively.

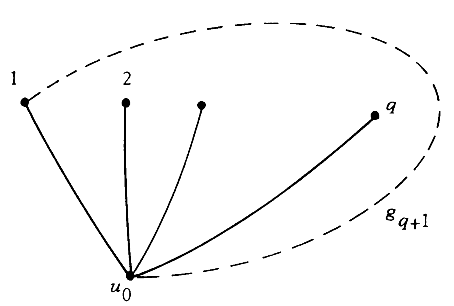

Figure 3

Hence, $D_{q+1} = \pm D_1 + b_2 D_2 + \cdots + b_q D_q$ for some integers b_i.

E. The precise determination of the b's depends on knowing the intersection numbers of the d_i. But we can still draw some conclusions about the P-L transformation without knowing them.

Clearly $D_{q+1} = D(d_{q+1}, h) \pm D_1$, where h is a loop in L enclosing all the branch points of this cycle. When t stays close to 0, the branch points and their cuts will stay inside this loop. Hence, h is an invariant path with respect to the change in t, so if T is the P-L transformation, we have

(6.2) **Proposition.** (1) $(T^q \pm 1)D_1 = D(T^{q+1}d_1, h)$,
(2) $T(D(T^q d_1, h)) = D(T^{q+1}d_1, h)$.

Thus, if $P(T)$ is the minimal polynomial for T on the homology of $H_{u0} \cap V_t$ (or just on d_1), we have

$$P(T)(T^q \pm 1)D_1 = 0. \tag{6.3}$$

We conclude that if Theorems I and II of the introduction are true in dimension $<\dim W$, they are true for the pencil V_t if the branch curve is nonsingular.

7. **Examples of branch curves.**

A. In general, we cannot assume that the branch curve is nonsingular or that the branch points move in circles. Even worse, although J is irreducible, it may not be so locally and several of the Puiseux series (6.1) may have the same center. So the cuts drawn to one cycle may interfere with those drawn to another of the same center. We give some examples of this behavior.

B. Consider the pencil of cubics

$$tx^3 + y^3 + z^3 = 0 \tag{7.1}$$

in the projective plane ($= W$). When $t = 0$, the fibre V_0 is $y^3 + z^3 = 0$, or three lines concurrent at $p = (1, 0, 0)$.

The general hyperplane pencil H_u is just the set of lines in W through some point. Choose the parameter u so that H_0 goes through p.

(7.2) **Proposition.** *$V_0^* = 6L_0$, where L_0 is the line in W^* representing the lines in W through p.*

(7.3) **Proof.** We calculate the dual of the general V_t. The generic point of V_t^* is (tx^2, y^2, z^2), where $tx^3 + y^3 + z^3 = 0$. If (A, B, C) are the coordinates in W^*, then V_t^* is given by $(tA)^{3/2} + B^{3/2} + C^{3/2} = 0$, which is equivalent to the polynomial relation

$$A^6 - 2tA^3(B^3 + C^3) + t^2(B^3 - C^3)^2 = 0. \tag{7.4}$$

When $t = 0$, we get the curve $A = 0$ counted 6 times. But $A = 0$ is just the line L_0 in W^*.

In order to find the Puiseux expansions around $t = 0$, we write the line H_u parametrically as

$$A = u, \quad B = au + b, \quad C = cu + d \tag{7.5}$$

and substitute in (7.1). The resulting expression in t and u is the local equation for J at $(0, 0)$.

One easily sees that

(7.6) $(0, 0)$ is a double point on J with double points in the first two neighborhoods. u is given by two Puiseux series $u_i(t) = e_i t^{1/3} + \cdots$ $(e_1 \neq e_2)$.

Here, the six branch points travel in noninterfering circles about 0, but undergo the permutation (135)(246) instead of (123456).

This kind of behavior allows the cuts to be generated by two of their number, i.e., we draw g_1 and g_2 as in part 6D (Figure 2) and observe that g_1 becomes $g_3, g_5, \cdots$ and g_2 becomes $g_4, g_6, \cdots$ as t moves about 0. Hence, the reasoning and conclusions of 6E are still valid.

Notice that all this goes through for any V_0 with an ordinary triple point off the base, since the question is local.

C. More generally, suppose V_0 is an s-fold curve (or component) with an ordinary r-fold point p.

Let x and y be affine coordinates in W at p and take the pencil V_t to be

$$(x^r + y^r)^s = t. \tag{7.7}$$

This special form will not affect the properties of the branch function, but makes calculation easier.

(7.8) **Proposition.** *For each V_t $(t \neq 0)$ there are $rs(r-1)$ branch points near p distributed in $r-1$ Puiseux series each varying like $t^{1/rs}$. The motion of the branch points and cuts is like the $(r-1)$st power at a simple branch point of order $rs(r-1)$, so the conclusions of* 6E *hold.*

(7.9) **Proof.** Let the lines H_u be given by $x = u$, $y = au + b$.

The branch curve is the locus where V_t and H_u are tangent, i.e.,

$$\operatorname{rank} \frac{\partial(V_t, H_u)}{\partial(x, y)} < 2.$$

The Jacobian here is

$$\begin{pmatrix} rs(x^r + y^r)^{s-1}x^{r-1} & rs(x^r + y^r)^{s-1}y^{r-1} \\ a & -1 \end{pmatrix}.$$

Hence, $x^{r-1} + ay^{r-1} = 0$, since $x^r + y^r \neq 0$ (we are not on V_0). The branch curve then consists of $r-1$ lines through p. Let $y = cx$ be one of them. Substituting in (7.7) gives $x^{rs} = c't$ for some constant $c' \neq 0$, or $u = c''t^{1/rs}$. This constant c'' is obviously different for each of the $r-1$ lines.

D. However, the branch points do not even have to move in a single circle. For example, take the case of plane quartics splitting into a cuspidal cubic plus a line through the cusp (or the corresponding local situation).

If the line is not the cuspidal tangent, we can take V_t to be

$$(y^2 - x^3)x = t. \tag{7.10}$$

(7.11) **Proposition.** *There are two Puiseux series centered at the cusp. One is* $t^{1/3}$ *and one is* $t^{1/4}$.

(7.12) **Proof.** Take the hyperplane pencil as in (7.9).

The Jacobian is

$$\begin{pmatrix} y^2 - 4x^3 & 2xy \\ a & 1 \end{pmatrix}$$

so the branch curve is

$$y^2 - 4x^3 - 2axy = 0.$$

It has an ordinary double point at the origin with tangents $y = 0$ and $y = 2ax$. The branch points on the $y = 0$ branch move like $t^{1/3}$ and those on $y = 2ax$ like $t^{1/4}$.

E. If the line through the cusp is the cuspidal tangent, our pencil becomes

$$(7.13) \qquad (y^2 - x^3)y = t$$

and the Jacobian is

$$\begin{pmatrix} -3x^2y & 3y^2 - x^3 \\ a & 1 \end{pmatrix}.$$

The branch curve is $3x^2y + 3ay^2 - ax^3 = 0$. At the origin, this is parametrically $x = 3s^2 + \cdots$, $y = s^3 + \cdots$, and when substituted in (7.13) becomes

$$-26s^9 + \cdots = t \text{ or } s = \text{P.S. } (t^{1/9}).$$

Then $x = c_1t^{2/9} + \cdots$, $y = c_2t^{3/9} + \cdots$ and $u = ac_1t^{2/9} + \cdots$. Here, the branch curve is locally irreducible.

8. Ordinary double points.

A. We apply the method of construction of homology to examine the case where V_0 has an ordinary double point off the base and demonstrate the results of 3C.

Choose the general hyperplane pencil H_u, $u \in L$. We can assume that no member of L is tangent to V_0 at points in the base of V_t or the base of L or at points where L is tangent to W.

(8.1) **Proposition.** *If* x *is a simple point of* V_0 *at which some* H_u *is tangent, then the Puiseux series* $u(t)$ *at* x *is holomorphic.*

(8.2) **Proof.** Let $z_1 = 0$ be the equation of V_0 at x, so the V_t is $z_1 = t$. If H_0 is tangent to V_0 at x, then H_u is given by $u = z_1 +$ h.p. Substituting t

for z_1 gives $u = h(t) + G$, where h is a power series in t and G is a power series in $t, z_2, \cdots, z_n$ of degree > 1 in the z's.

This is the local equation (in V_t) of $V_t \cap H_u$. Hence, it has a singular point near x if and only if $u = h(t)$, so u is indeed holomorphic in t.

(8.3) **Proposition.** *The cone over the path from* u_0 *to* $u(t)$ *is invariant.*

(8.4) **Proof.** (8.1) says that the path is invariant. On the other hand the vanishing cycle over the path is invariant, since $V_0 \cap H_{u0}$ is nonsingular in the pencil $V_t \cap H_{u0}$.

B. The only point left to consider is the double point p itself.

(8.5) **Proposition.** *The branch curve is nonsingular at* p *and* u *is a power series in* $t^{1/2}$.

(8.6) **Proof.** By 4A, we can take the pencil to be $z_1^2 + z_2^2 + \cdots + z_n^2 = t$.

Let H_u be $u = \Sigma c_i z_i$ + h.p., so that H_0 goes through p. The hyperplane pencil being general, we have that H_0 is not tangent to the tangent cone at p in V_0, i.e., $\Sigma c_i^2 \neq 0$.

The Jacobian is

$$\begin{pmatrix} 2z_1 & \cdots & 2z_n \\ c_1 + \cdots & & c_n + \cdots \end{pmatrix}.$$

When its rank is 1, we have

$$z_i = \lambda(c_i + \cdots) \tag{8.7}$$

for some λ and all i. We are only interested in small λ, since the branch points we want are close to p.

The coefficient matrix of the linear part of (8.7) is of the form $I - \lambda M$, where I is the identity $n \times n$ matrix and M is the Hessian of H_0 at p. Hence, $\det(I - \lambda M) \neq 0$ for small λ, which means that (8.7) can be uniquely inverted near p to yield the branch curve whose parametric equations are $z_i = c_i\lambda + \cdots$, all i. Hence, $t = \Sigma c_i^2\lambda^2$ + h.p. and $u = \Sigma c_i^2\lambda$ + h.p. and u is a power series in $t^{1/2}$.

Note that we could have used the simpler pencil $u = z_2$. The branch function is then just $u = t^{1/2}$. This device of using simpler local equations will be helpful later on.

C. Draw the cuts g_1 and g_2 as in 6D and then any others needed for the other branch points.

If an element D of $H_{n-1}(V_t, V_t \cap H_{u0})$ is written in the form $D = q_1 D_1 + q_2 D_2 + \cdots$, then by (8.3), $(T - 1)D = q_1(T - 1)D_1 + q_2(T - 1)D_2$. We have $T(D_1) = D_2$ (by 6D). What is $T(D_2)$?

There are two cases, depending on whether n is even or odd.

(8.8) **Proposition.** *If n is odd, $T(D_2) = 2D_2 - D_1$. If n is even, $T(D_2) = D_1$.*

(8.9) **Proof.** We use induction. Let d vanish along g_1 and generate D_1. Then d also generates D_2. If n is odd, then (3.7) says that d goes to $-d$ after u goes around g_2, while if n is even d is invariant around g_2. In either case (8.8) follows.

The discussion of part 4D applies to V_0, as can be seen by blowing up its singular point and noting that the pencil $H_u \cap \overline{V}_0$ is still sufficiently general in the sense of Lefschetz (see (3.8)). The section $H_0 \cap V_0$ has an ordinary double point, so as $H_u \cap V_0$ approaches $H_0 \cap V_0$, a single cycle will vanish and trace out a cone D_0 on V_0. The path of approach from u_0 to 0 (in L) can be taken to be g_0, near g_1 and g_2.

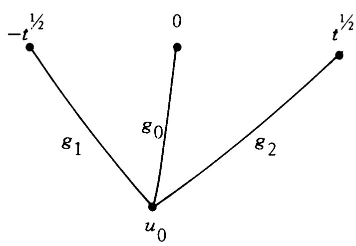

Figure 4

Hence, as $t \to 0$, d will approach d_0 and D_1 and D_2 will become D_0. In other words, $D_1 - D_2$ is an $(n-1)$-cycle on V_t which vanishes at V_0. Since D_1 and D_2 are cones over a sphere, $D_1 - D_2$ is topologically a sphere, too. (8.8) then yields

(8.10) If n is odd, $T(D_1 - D_2) = -(D_1 - D_2)$. If n is even, $T(D_1 - D_2) = D_1 - D_2$.

This verifies most of 3C except for the intersection numbers.

D.

(8.11) **Proposition.** *$D_1 - D_2$ can be pulled off any algebraic subset A of V_t.*

(8.12) **Proof.** Move the base-point u_0 slightly. Then d, D_1 and D_2 will vary continuously into $\overline{d}$, $\overline{D}_1$ and $\overline{D}_2$.

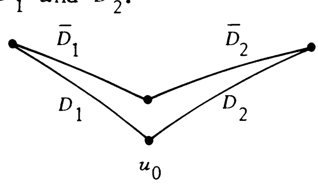

Figure 5

Hence, $D_1 - D_2 = \bar{D}_1 - \bar{D}_2$ in $H_{n-1}(V_t)$. Since by induction d can be pulled off $V_t \cap H_u \cap A$ for each u (except perhaps the singular points), we need only choose the pencil L so that its tangencies on V_t miss A. Hence, all of $\bar{D}_1 - \bar{D}_2$ is disjoint from A.

This also shows that $D_i \cap (\bar{D}_1 - \bar{D}_2) = \emptyset$ if $i > 2$, so we need only check (3.6) for D_1 and D_2.

We have $T(D_1) = D_1 - (D_1 - D_2)$. Hence, we must show that $(D_1, D_1 - D_2) = (-1)^{(n-1)(n-2)/2}$.

This intersection number is the same as $(D_1, \bar{D}_1 - \bar{D}_2)$ which is also $(D_1, \bar{D}_1)$.

(8.13) $(D_1, \bar{D}_1) = (-1)^{(n-1)(n-2)/2}$.

(8.14) **Proof.** Choose coordinates $z_1, \cdots, z_n$ at the vertex of D_1 so that $H_u \cap V_t$ is given in V_t by $z_1^2 + \cdots + z_{n-1}^2 = u$, and the cone $\{z_i \text{ real} \mid \Sigma z_i^2 \leq 1\}$ is the tip of D_1.

$\bar{D}_1$ is obtained by moving u slightly counterclockwise from 1 and is given by

$$\left\{(z_1, \cdots, z_{n-1}) \,\middle|\, z_1 u^{-1/2} \text{ real and } \sum (z_i u^{-1/2})^2 \leq 1\right\}.$$

Hence, $\bar{D}_1 = u^{1/2} D_1$.

Using Lefschetz's prescription for calculating intersection numbers [4], let $(P_0, \cdots, P_{n-1})$ be an orientation for the cell D_1 where $P_0 = (0, \cdots, 0)$. Then the orientation for $\bar{D}_1$ is $(P_0, u^{1/2}P_1, \cdots, u^{1/2}P_{n-1})$.

Putting these together gives the orientation $(P_0, P_1, \cdots, P_{n-1}, u^{1/2}P_1, \cdots, u^{1/2}P_{n-1})$ for V_t near P_0. But the orientation V_t has as a complex manifold is $(P_0, P_1, iP_1, \cdots, P_{n-1}, iP_{n-1}) = (P_0, P_1, u^{1/2}P_1, \cdots, P_{n-1}, u^{1/2}P_{n-1})$, since $u^{1/2}$ is counterclockwise from 1. To change the first orientation to the second requires $(n-2) + (n-3) + \cdots + 2 + 1 + 0$ transpositions, so $(D_1, \bar{D}_1) = (-1)^{(n-1)(n-2)/2}$.

Similarly, (3.6) can be checked for D_2, since $(\bar{D}_2, D_2) = (-1)^{(n-1)(n-2)/2}$.

(8.15) **Corollary.** (3.7) *holds*.

(8.16) **Proof.** When n is even, this is clear, since $D_1 - D_2$ is then odd-dimensional and so always has self-intersection number 0.

When n is odd, $(D_1 - D_2, D_1 - D_2) = (D_1 - D_2, \bar{D}_1 - \bar{D}_2) = (D_1, \bar{D}_1) + (D_2, \bar{D}_2) = 2(-1)^{(n-1)(n-2)/2}$. But $(n-1)(n-2)/2 \equiv (n-1)/2 \pmod 2$, when n is odd.

9. Normal crossings.

A. Suppose there is no base in the pencil V_t, so f is everywhere defined. The fibre V_0 can have arbitrary singularities, but we can use Hironaka's results on the resolution of singularities [2] which say that by repeated monoidal transformations centered in V_0, we can transform W to a nonsingular $\overline{W}$ with a birational morphism $g\colon \overline{W} \longrightarrow W$ such that

(9.1) (1) g is an isomorphism between $\overline{W} - g^{-1}(V_0)$ and $W - V_0$.

(2) $g^{-1}(V_0)$ is a divisor on $\overline{W}$ with only normal crossings as singularities (see below for definition).

Now $h = f \circ g$ is a morphism from $\overline{W}$ to C and so induces a pencil of fibres $\overline{V}_t$ in $\overline{W}$. By (9.1), $\overline{V}_0 = g^{-1}(V_0)$ and $\overline{V}_t \simeq V_t$, $t \neq 0$. W and $\overline{W}$ are the same fibre space over $C - \{0\}$, so the P-L transformation on V_t is the same as that on $\overline{V}_t$.

Hence, we can assume from the start that V_0 has only normal crossings.

B.

(9.2) **Definition.** The divisor V_0 is said to consist of *normal crossings* if its support is the union of nonsingular hypersurfaces of W which are transversal at every point.

This means that if $V^1, \cdots, V^s$ are the components of V_0 containing $x \in V_0$ and if $z_i = 0$ $(i = 1, \cdots, s)$ is a local equation for V^i at x, then the z_i are a partial system of coordinates at x in W.

Suppose that all the components of V_0 are $V^1, \cdots, V^r$ and that V^i has multiplicity $m_i > 0$. Then near x, the pencil V_t can be given by $z_1^{m_1} \cdots z_s^{m_s} = t$. x is then $(m_1 + \cdots + m_s)$-fold singular for the divisor V_0 and s-fold singular for its support $|V_0|$. Conversely, a point which is s-fold on $|V_0|$ belongs to exactly s components of V_0.

Let M_i = set of points i-fold singular for $|V_0|$ $(i = 1, \cdots, n)$. The M_i possess the following properties:

(9.3) (1) They are disjoint.

(2) $\bigcup_i M_i = |V_0|$.

(3) If $M_i \neq \emptyset$, it is a nonsingular locally closed algebraic subset of W of dimension $n - i$.

(4) $\partial M_i = M_{i+1} \cup \cdots \cup M_n$.

(5) Each component of $\overline{M}_i$ is nonsingular.

C. Thus, we have a natural partition of V_0 into its singular subvarieties. It is clear what V_0^* should be, viz., those hyperplanes tangent to some component of some $\overline{M}_i$. We shall show this later.

We first need some information about general hyperplane pencils.

(9.4) **Proposition.** *Let $V_1, V_2, \cdots$ be a finite number of nonsingular irreducible subvarieties of P_N and $A_i \subseteq V_i$ proper algebraic subsets. Then in a general pencil of hyperplanes, the points of tangency on V_i do not lie on A_i.*

(9.5) **Proof.** Each V_i^* is an irreducible subvariety in P_N^* and the hyperplanes tangent to V_i at a point of A_i form a proper algebraic subset A_i^* of V_i^*. Hence, the general line in P_N^* misses every A_i^*.

We apply this to the collection of components of the $\overline{M}_i$. Let the A's be the parts of these components in ∂M_i. (9.4) then yields

(9.6) **Corollary.** *Let H_u be a general pencil of hyperplanes. Then if H_u is tangent to a component of $\overline{M}_i$, the point of tangency lies in M_i.*

Furthermore, those hyperplanes tangent to more than one of these components form a subset of P_N^* of dimension $< N-1$, so the pencil L misses it. Hence,

(9.7) Each H_u is tangent to at most one component of the $\overline{M}_i$ and the tangent section will have a single ordinary double point not lying on the base locus $B(L)$ of the pencil H_u. In particular, no point of M_n is in $B(L)$.

D. We now examine the branch curve for V_t.

(9.8) **Proposition.** *Let $x \in V_0 \cap B(L)$. Then x is not on the branch curve.*

(9.9) **Proof.** Let x belong to $V^1, \cdots, V^s$, so V_t is $z_1^{m_1} \cdots z_s^{m_s} = t$ near x. Now $s < n$, or else $x \in M_n \cap B(L)$. M_s is given locally by $z_1 = \cdots = z_s = 0$. If $h = 0$ is a local equation for one of the hyperplanes H, then since $H \cap M_s$ has no singularity at x, the functions $z_1, \cdots, z_s, h$ form a partial system of coordinates there. Let us rename h and call it z_{s+1}. Complete to a full set of coordinates $z_1, \cdots, z_n$.

Let $h' = c_i z_i + \text{h.p.} = 0$ be the local equation for another hyperplane in L. For some $i \neq s+1$, $c_i \neq 0$. Otherwise, $h' - c_{s+1}z_{s+1} = 0$ is a hyperplane in L tangent to W at $x \in B(L)$, which contradicts the generality of L. Also, $i > s$, else $h' = 0$ would be tangent to M_s at x.

$H \cap V_t$ is given by the same equation $z_1^{m_1} \cdots z_s^{m_s} = t$ in V_t, so H is not tangent to V_t, $t \neq 0$. Every other hyperplane can be written in the form $h' + uz_{s+1}$. Hence, the Jacobian is

$$\begin{pmatrix} m_1 z_1^{m_1-1} \cdots z_s^{m_s} & \cdots & m_s z_1^{m_1} \cdots z_s^{m_s-1} & 0 & 0 \cdots & 0 \\ * & \cdots & * \cdots & * & c_i + \text{h.p.} & * \end{pmatrix}.$$

When $t \neq 0$, none of $z_1, \cdots, z_s$ is zero, while near x, $c_i + \text{h.p.}$ will be nonzero. Hence, this Jacobian has rank 2 and x cannot be on the branch curve.

(9.10) **Proposition.** *Suppose $x \notin B(L)$ but no H_u is tangent to M_s at x. Then x is not on the branch curve.*

(9.11) **Proof.** The proof is the same as (9.9) except that now the hyperplane pencil takes the simpler form $u = z_{s+1}$. The Jacobian is

$$\begin{pmatrix} m_1 t/z_1 & \cdots & m_s t/z_s & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

so is of rank 2.

E.

(9.12) **Proposition.** *Suppose $x \notin B(L)$, $x \in M_s$ and some hyperplane in L is tangent to M_s at x. Then,*

(1) *x belongs to the branch curve.*

(2) *If V_t is $z_1^{m_1} \cdots z_s^{m_s} = t$ and $m = m_1 + \cdots + m_s$, then u is a Puiseux series in $t^{1/m}$.*

(3) *x is nonsingular on the branch curve and u is a uniformizing parameter there.*

(9.13) **Proof.** Let H_u be $u = \sum c_i z_i +$ h.p. near x. If H_0 is tangent to M_s at x, then $c_{s+1} = \cdots = c_n = 0$.

If some c_i $(i \leq s)$ were zero, then H_0 would be tangent to some $M_{s'}$ $(s' < s)$ at x. This contradicts (9.6). So none of $c_1, \cdots, c_s$ is zero.

The Jacobian is

$$\begin{pmatrix} m_1 z_1^{m_1 - 1} \cdots z_s^{m_s} & \cdots & m_s z_1^{m_1} \cdots z_s^{m_s - 1} & 0 & 0 & \cdots & 0 \\ c_1 + \text{h.p.} & \cdots & c_s + \text{h.p.} & \dfrac{\partial u}{\partial z_{s+1}} & & \cdots & \dfrac{\partial u}{\partial z_n} \end{pmatrix}.$$

When $t \neq 0$, neither are $z_1, \cdots, z_s$, so the Jacobian has the same rank as

$$\begin{pmatrix} 1 \quad 1 \quad 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \dfrac{c_1 z_1}{m_1} + \text{h.p.} & \cdots & \dfrac{c_s z_s}{m_s} + \text{h.p.} & \dfrac{\partial u}{\partial z_{s+1}} & & \cdots & \dfrac{\partial u}{\partial z_n} \end{pmatrix}.$$

Hence, the points on the branch curve near x are those satisfying the simultaneous equations

(9.14) $\lambda = c_i z_i / m_i +$ h.p. $(i = 1, \cdots, s)$ and $\partial u / \partial z_j = 0$ $(j = s+1, \cdots, n)$ for some (small) λ.

x satisfies these equations when $\lambda = 0$, which establishes (1). For (2) and (3), we must solve for the z's in terms of λ.

The Jacobian of (9.14) at x is

$$\begin{pmatrix} c_1/m_1 & \cdots & 0 & \\ & & & 0 \\ 0 & \cdots & c_s/m_s & \\ & 0 & & K \end{pmatrix},$$

where K is the Hessian of u with respect to $z_{s+1}, \cdots, z_n$ evaluated at x. Hence, its determinant is

$$\frac{c_1 \cdots c_s}{m_1 \cdots m_s} \det K.$$

We already know that no $c_i = 0$. But K is also the Hessian of $H_0 \cap M_s$ at x in M_s, and since x is an ordinary double point for $H_0 \cap M_s$, its determinant is nonzero.

It follows that (9.14) can be inverted near x to yield

(9.15) $z_i = d_i\lambda + \text{h.p.}$ $(d_i = m_i/ c_i,\ i = 1, \cdots, s)$ which are parametric equations for the branch curve. Substituting (9.15) in the expressions for u and t, we get $u = m\lambda + \text{h.p.}$ and $t = c\lambda^m + \text{h.p.}$, $c \neq 0$. This gives (2) and (3).

(9.16) *Note* 1. We could have simplified the calculations had we taken H_u to be, say, $z_1 + \cdots + z_s + z_{s+1}^2 + \cdots + z_n^2 = u$, since this variety is also tangent to no $M_{s'}$ $(s' < s)$ and cuts an ordinary double point on M_s.

Note 2. Since all the calculations are local, (9.12) also holds for those V_0 which consist of *local* normal crossings, i.e., such that locally V_0 is $z_1^{m_1} \cdots z_s^{m_s} = 0$ for some local coordinates z_i.

F. The considerations of part 6E apply to the case of normal crossings since the branch curve is nonsingular at $t = 0$ and the general section $H_{u0} \cap V_0$ also has normal crossings. We know Theorems I and II ((1.6) and (1.8)) hold when $\dim W = 1$ (part 2A).

Hence,

(9.17) Theorems I and II are established for arbitrary dimension.

(9.17) is equivalent to

(9.18) There is an integer N such that the P-L transformation satisfies the relation $(T^N - 1)^n = 0$, $n = \dim W$.

10. The main theorems.

A. In order to prove Theorems I′ and II′ (part 1C) we must make a closer analysis of the behavior of the cones.

As in the previous part, consider the m cones corresponding to a point x on M_s and V_t: $z_1^{m_1} \cdots z_s^{m_s} = t$. With respect to L we have m cuts g_i, vanishing cycles d_i and cones D_i. If T is the P-L transformation and $T^m(D_1) = q_1 D_1 + \cdots + q_m D_m$, then $P(T) = T^m - q_m T^{m-1} - \cdots - q_i$ is the minimal polynomial of T on this group of cones (considered as elements of $H_{n-1}(V_t - H_\infty, V_t H_{u0} - H_\infty)$, cf. (3.17)).

Theorems I′ and II′ follow immediately from

(10.1) **Theorem.** $P(T) = \Pi\,(T^{m_i} - 1)$.

(10.2) **Proof.** The method of proof is to take a new hyperplane pencil L' with which we can express the homology of $V_t \cap H_u$ and hence the vanishing cycles d_i. There are two main cases, according as $s < n$ (i.e., dim $M_s > 0$) or $s = n$ (i.e., x is one of the points of M_n).

B. *Case* 1. $s < n$. First we remark that the base section u_0 can be chosen close enough to 0 so as to meet any pre-assigned neighborhood of x, for instance, the domain of definition of the local coordinates $z_1, \cdots, z_n$, but not close enough to interfere with the $u(t)$, t small.

For convenience, let L be given by $u = z_1 + \cdots + z_s + z_{s+1}^2 + \cdots + z_n^2$. The lack of higher powers will not affect the conclusions (see note 1 of 9E). The new general pencil L' can be taken to be $z_{s+1} =$ const, since no member of it is tangent to M_s near x.

(10.3) **Proposition.** (1) *The pencil L' has 2m tangencies on $V_t \cap H_{u0}$ near x.*

(2) *The 2m branch points are arranged in two nonintersecting circles of m points each (when t is small), each moving as around a point of type $(m_1, \cdots, m_s)$ when t goes about* 0.

(3) *If the cuts in these two groups are chosen appropriately, then as u approaches one of the $u(t)$, the cycle vanishing there is the sum of a cone from one group and a cone from the other.*

(10.4) **Proof.** The tangencies of L' on $V_t \cap H_{u0}$ depend on the places where L' is tangent to the singularities of $V_0 \cap H_{u0}$ (part 9E). Near x there are only s components for both V_0 and $V_0 \cap H_{u0}$; hence, there will be the singular subvarieties $M_1', \cdots, M_s'$ on $V_0 \cap H_{u0}$ (analogous to the M_i on V_0). In fact, $M_i' = M_i \cap H_{u0}$ and points in each are of the same type.

Now H_u is not tangent to M_i $(i < s)$ near x, so no member of L' can be tangent to $M_i \cap H_{u0}$ (because $M_i \cap H_u$ is a pencil without singularities). But since H_0 cuts an ordinary double point on M_s, a nearby section, like $M_s \cap H_{u0}$, will have two tangent planes in L' (part 8). Hence, for small t, there will be two groups of branch points on $V_t \cap H_{u0}$.

Since $V_0 = m_1 V^1 + \cdots + m_s V^s$ near x, $H_0 \cap H = m_1 V^1 \cap H_{u0} + \cdots + m_s V^s \cap H_{u0}$, so both these groups are of type $(m_1, \cdots, m_s)$ and there are m points in each.

As u_0 approaches one of the $u(t)$, two branch points come together, one from each group, so the vanishing cycle is the sum of a cone from each group.

We can demonstrate (10.3) analytically by using the local equations for L and L':

V_t, H_u and $z_{s+1} = 0$ meet in a singular variety when the Jacobian matrix $\partial(V_t, H_u, z_{s+1})/\partial(z_1, \cdots, z_n)$ has rank < 3.

Here, this Jacobian is

$$\begin{pmatrix} m_1 z_1^{m_1 - 1} \cdots z_s^{m_s} & \cdots & m_s z_1^{m_1} \cdots z_s^{m_s - 1} & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & & 2z_{s+1} & \cdots & 2z_n \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Since no $z_i = 0$ $(i = 1, \cdots, s)$, the rank of this matrix is the same as that of

$$\begin{pmatrix} m_1 & m_2 & \cdots & m_s & 0 & 0 & \cdots & 0 \\ z_1 & z_2 & \cdots & z_s & 2z_{s+1} & 2z_{s+2} & \cdots & 2z_n \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Rank < 3 means that $z_i = \lambda m_i$ $(i = 1, \cdots, s)$ and $z_{s+2} = \cdots = z_n = 0$. Hence, $u = m\lambda + z_{s+1}^2$ and $t = c\lambda^m$, $c \neq 0$. Hence, the branch points z_{s+1} on $V_t \cap H_u$ are given by $z_{s+1} = (u - c' t^{1/m})^{1/2}$, $c' \neq 0$.

The assertions of (10.3) follow easily from the simple nature of this algebraic function.

C. We apply (10.3) to prove (10.1).

Consider the boundary map

$$\partial\colon H_{n-1}(V_t, V_t \cap H_{u_0}) \longrightarrow H_{n-2}(V_t \cap H_{u_0}, V_t \cap H_{u_0} \cap H'_{v_0})$$

where H'_{v_0} is in L'. Better still, consider the same map on the relative groups of the corresponding affine varieties (i.e., with the hyperplane at infinity removed).

By (3.17) the cones we have been considering are free generators of their respective groups. We have $P(T)D_1 = 0$. Applying ∂ gives $P(T)d_1 = 0$. But (10.3) tells us that d_1 is the sum of cones on which, by induction, the P-L transformation has minimal polynomial $Q(T) = \prod (T^{m_i} - 1)$.

Hence, $Q(T)$ divides $P(T)$, but since they are both monic and of the same degree m, they must be equal.

D. *Case* 2. $s = n$. In this case there will be no nice form for the new pencil L', since we have run out of coordinates. The best we can do is let L be $u = z_1 + \cdots + z_n$, as before, and L': $v = c_1 z_1 + \cdots + c_n z_n$, with distinct nonzero constants c_i.

Let $V_0 = m_1 V^1 + \cdots + m_n V^n$ near x. We need to know the tangencies of L' on $V_t \cap H_{u_0}$.

(10.5) **Proposition.** (1) *L' has $(n-1)m$ tangencies on $V_t \cap H_{u_0}$ near x.* (2) *These $(n-1)m$ branch points are arranged in n nonintersecting circles* *of* $m - m_i$ *points each (when* t *is small), each moving as around a point of type* $(m_1, \cdots, \hat{m}_i, \cdots, m_n)$ *when* t *goes around* 0.

(10.6) **Proof.** As in part B we examine the singular subvarieties of $V_0 \cap H_{u0}$. Since this is $(n-2)$-dimensional, we have the partition (near x) $V_0 \cap H_{u0} = M_1' \cup \cdots \cup M_{n-1}'$ where $M_i' = M_i \cap H_{u0}$. As before, the H_u (u small) are not tangent to $M_1, \cdots, M_{n-1}$, so L' will not have tangents on $M_1', \cdots, M_{n-2}'$ near x (because their dimension is > 0 and L' is general).

But M_{n-1}' is a set of points, so the members of L' we seek are the ones that contain a point of M_{n-1}'.

Let p_i be the point $(0 \cdots u_0 \cdots 0)$, with u_0 in the ith coordinate. This is where $V^1, \cdots, \hat{V}^i, \cdots, V^n, H_{u0}$ meet near x. Now $V_0 \cap H_{u0} = m_1(V^1 \cap H_{u0}) + \cdots + m_n(V^n \cap H_{u0})$ and $M_{n-1}' = \{p_i\}$. We now just remark that p_i is of type $(m_1, \cdots, \hat{m}_i, \cdots, m_n)$ and so contributes a cycle of $m - m_i$ branch points. Finally, the total number of branch points near x is $\Sigma(m - m_i) = (n-1)m$.

The picture in the v-line L' looks like

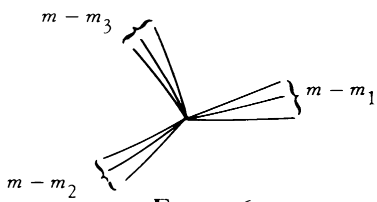

Figure 6

E. We must now examine the homology groups of the several varieties at hand. Since the question is local, the topology involved is the same as that of the affine pencil $z_1^{m_1} \cdots z_n^{m_n} = t$. The topology of V_t is given by

(10.7) If k is the g.c.d. of the m_i, then V_t is homeomorphic to the sum of k copies of $T^{n-1} \times R^{n-1}$, where T^{n-1} is an $(n-1)$-torus and R^{n-1} is Euclidean $(n-1)$-space. The k pieces of V_t permute cyclically as t circles 0.

(10.8) **Proof.** One just examines the map λ: $V_t \longrightarrow R^{n-1}$ given by $\lambda(z_1, \cdots, z_n) = (|z_1|, \cdots, |z_n|)$ and observes that the image is homeomorphic to R^{n-1} and each fibre is k copies of T^{n-1}.

Now consider the exact sequence

(10.9) $H_{n-1}(V_t \cap H_{u0}) \longrightarrow H_{n-1}(V_t) \longrightarrow H_{n-1}(V_t, V_t \cap H_{u0}) \xrightarrow{\partial} H_{n-2}(V_t \cap H_{u0})$.

The first group is 0 (3.15), the second is $\mathbf{Z}^k$ and the third is $\mathbf{Z}^m$. Since the D_i generate $H_{n-1}(V_t, V_t \cap H_{u0})$ and $\partial D_i = d_i$, the d_i span a subgroup of rank $m - k$ in $H_{n-2}(V_t \cap H_{u0})$. In other words, there are just k relations among them.

Let $P(T)$ be the minimal polynomial of T on the D's and $Q(T)$ the minimal polynomial on the d's. Then $\deg p = m$ and $Q(T) \mid P(T)$. Now the minimal polynomial of T on *all* the $(n-1)m$ cones in $V_t \cap H_{u0}$ is

$$R(T) = \underset{i}{\text{l.c.m.}}\ ((T^{m_1} - 1), \cdots, \widehat{(T^{m_j} - 1)}, \cdots, (T^{m_n} - 1))$$

$$= \prod (T^{m_i} - 1)/(T^k - 1).$$

Then $Q(T) \mid R(T)$, so $\deg Q \leq \deg R = m - k$. If $\deg Q < m - k$, then all the d's would depend on the first $m - k - 1$: $d_1, T(d_1), \cdots, T^{m-k-2}(d_1)$, a contradiction. Hence, $Q = R$.

Thus, $P = RS$ where S is monic and of degree k.

Now observe that because the k pieces of V_t are disjoint and permuted cyclically by T, the cones $D_i, T(D_i), \cdots, T^{k-1}(D_i)$ are all disjoint for any i. Hence, when we look at $D(d_{m+1}, g_{m+1})$ and bring d_{m+1} back around each g_i to calculate the coefficients of P, it will never involve a D_i unless $i \equiv m \pmod{k}$. Hence, $P(T)$ is a polynomial in T^k, so $S(T) = T^k \pm 1$. To prove (10.1), we need $S(T) = T^k - 1$.

Consider the loop b which surrounds all m branch points in L. After carrying $d_{m+1} = T^m(d_1)$ around b, it vanishes along g_1. Hence, it becomes $\pm d_1$ after traversing b. The following proposition is easily verified.

(10.10) **Proposition.** *d_{m+1} becomes d_1 if and only if $S(T) = T^k - 1$.*

Hence, we assume (10.10) holds in dimension $< n$ and allow u to trace out b, keeping t small. The m points M'_{n-1} trace out disjoint circles on M_{n-1}, so the m groups of branch points in $V_t \cap H_{u0}$ merely rotate around $v = 0$ but within each group come back to their original positions in a different order.

(10.11) **Proposition.** *Each of the m cycles undergoes the transformation T^{-m}.*

(10.12) **Proof.** Let us look at the point $p_n = (0\ 0 \cdots 0\ u)$ in $M'_{n-1}(u)$. If L' is $v = c_1 z_1 + \cdots + c_{n-1} z_{n-1} + z_n$, H'_v goes through p_n for $v = u$. Then we know by (9.12) that $v = u + ct^{1/(m-m_n)} + \cdots$, c depending on u.

We calculate $c(u)$ as follows:

On H_u, the pencil $V_t \cap H_u$ is $z_1^{m_1} \cdots z_{n-1}^{m_{n-1}}(u - z_1 - \cdots - z_{n-1})^{m_n} = t$ and L' becomes $v = u + (c_1 - 1)z_1 + \cdots + (c_{n-1} - 1)z_{n-1}$. In order to make the equation for $V_t \cap H_u$ have the simple form of (9.12), replace z_{n-1} by $\bar{z}_{n-1} = z_{n-1}(u - z_1 - \cdots - z_{n-1})^{m_n/m_{n-1}}$, so $V_t \cap H_u$ is given by $z_1^{m_1} \cdots z_{n-2}^{m_{n-2}} \bar{z}_{n-1}^{m_{n-1}} = t$.

Hence, the linear part of v is $u + (c_1 - 1)z_1 + \cdots + (c_{n-2} - 1)z_{n-2} + (c_{n-1} - 1)\bar{z}_{n-1}(u - z_1 - \cdots - z_{n-1})^{-m_n/m_{n-1}}$, or, when t is small enough,

$$u + (c_1 - 1)z_1 + \cdots + (c_{n-2} - 1)z_{n-2} + (c_{n-1} - 1)u^{-m_n/m_{n-1}}\bar{z}_{n-1}.$$

From (9.12) we get

$$z_i = (m_i/(c_i - 1))\lambda + \cdots, \qquad i = 1, \cdots, n-2,$$

and

$$\bar{z}_{n-1} = (m_{n-1}/(c_{n-1} - 1))u^{m_n/m_{n-1}}\lambda + \cdots.$$

This means $v - u = m\lambda + \cdots$ and

$$t = c\lambda^{m-m_{n-1}-m_n}\lambda^{m_{n-1}}u^{m_n} + \cdots = c\lambda^{m-m_n}u^{m_n} + \cdots, \qquad c \text{ a constant.}$$

Hence, $v = u + c'u^{-m_n/(m-m_n)}t^{1/(m-m_n)} + \cdots$, c' a constant, or $v = u(1 + c'(u^{1/(m-m_n)})^{-m}t^{1/(m-m_n)} + \cdots)$.

So as u travels on h (t fixed), the Puiseux series around p_n gets multiplied by ϵ^{-m}, where ϵ is a primitive $(m - m_n)$th root of 1.

Hence, the $m - m_n$ cones in this group undergo the transformation T^{-m}. Similarly for all the other groups, which proves (10.11).

Since, d_1 is a sum of these $(n-1)m$ cones, (10.10) implies that $P(T) = \Pi\,(T^{m_i} - 1)$, as desired.

11. Normal crossings at the base.

A. We now weaken the assumptions we have been working under to allow for a base locus. This means that the parameter curve C is the projective line P^1, so that the V_t are just a 1-dimensional linear system on W. Any two members will meet in the same algebraic cycle B of codimension 2 in W.

The singularities of B coincide with the points where some V_t has singularities at the base. This is the same as the set of points where two (nonsingular) V_t are tangent. A singularity of B is singular for a unique V_t, since we are still assuming that the general fibre is nonsingular.

The homeomorphisms between the nonsingular V_t induced by the local retractions clearly leave B point-wise fixed and the same holds for any homology cycle that can be put in B. This, of course, is why there is no action in homology (outside of the middle dimension) when the V_t are hyperplane sections.

In theory, one can eliminate the base by the following device:

Our family V_t is given by the rational map $f\colon W \to P^1$. Let W' be the graph of f in $W \times P^1$. Then we have the rational maps $f'\colon W' \to P^1$ and $g\colon W' \to W$ induced by the projections. We have

(11.1) (1) f' and g are everywhere defined.

(2) $f \circ g = f'$.

(3) g is a biregular isomorphism between $W' - g^{-1}(B)$ and $W - B$.

(4) If $t \in P^1$, then $g\,|\,f'^{-1}(t)$ is an isomorphism of $f'^{-1}(t)$ with V_t.

By (3), the only singularities of W' lie in $g^{-1}(B)$, so if W'' is the desingularization (Hironaka) of W' obtained by blowing up pieces of $g^{-1}(B)$ and if g': $W'' \to W'$ is the associated map, we get a rational map $f' \circ g' = f \circ g \circ g'$ which is defined everywhere on W'' and gives us a pencil V_t' without base. If we consider the map $g \circ g'$: $W'' \to W$, then $g \circ g'$: $V_t' \to V_t$ is a blowing-up of V_t along pieces of B.

This means that the map $H_*(V_t') \to H_*(V_t)$ induced by $g \circ g'$ is surjective. Hence, if the main theorems hold for V_t', they hold for V_t.

B. Still, it may be worthwhile to examine the pencil V_t directly when the base B is particularly simple. This is useful for hypersurfaces in projective space, where a base locus cannot be avoided.

As before, we assume the singular fibre V_0 has only normal crossings. We will also assume that the pencil V_t is *transversal at the base*. This just means that the components of V_0 and any one of the other V_t are transversal wherever they meet, i.e., their local equations form a partial system of coordinates. This implies that only V_0 has singularities on B.

Analytically, this means that if $x \in B$ and $V^1, \cdots, V^s$ are the components of V_0 through x with local equations $z_1, \cdots, z_s$ and multiplicities $m_1, \cdots, m_s$, then $s < n$ and there is another local coordinate z_{s+1} such that V_t is given by $z_1^{m_1} \cdots z_s^{m_s} = tz_{s+1}$.

Hence, the base (near x) is given by

$$(z_1 = z_{s+1} = 0) \cup (z_2 = z_{s+1} = 0) \cup \cdots \cup (z_s = z_{s+1} = 0).$$

Hence, B consists of normal crossings, i.e., V_t $(t \neq 0)$ is not tangent to any M_i. Let $B_i = M_i \cap B$.

C. We now examine V_0^*. (9.4) and (9.6) yield

(11.2) **Proposition.** *Let H_u be a general pencil of hyperplanes. Then each H_u is tangent to at most one of the M_i or B_i and cuts an ordinary double point there. Also, if H_u is tangent to $\overline{M}_i$ (or $\overline{B}_i$), it is tangent to M_i (resp. B_i) but not at B.*

The branch function and P-L transformation for tangencies of H_u with M_i were given by (9.12). For a tangency at B, we have

(11.3) **Proposition.** *If $x \in B_s$ and V_t is $z_1^{m_1} \cdots z_s^{m_s} = tz_{s+1}$ near x and if some H_u is tangent to B_s at x, then x is on the branch curve and the branch function is given by $u =$ Puiseux series in $t^{1/(m-1)}$, $m = m_1 + \cdots + m_s$.*

(11.4) **Proof.** We can take H_u to be $z_1 + \cdots + z_{s+1} = u$. The Jacobian is

$$\begin{pmatrix} m_1 t z_{s+1}/z_1 & \cdots & m_s t z_{s+1}/z_s & -t & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

When the rank is < 1, we find $z_i = m_i z_{s+1}(-1)$ $(i = 1, \cdots, s)$ or $cz^m_{s+1} = tz_{s+1}$ and $u = -(m-1)z_{s+1}$, from which (11.3) follows.

Hence, for each tangency on B_s we have a cycle of $m-1$ branch points, so the P-L transformation on this group is a polynomial of degree $m-1$. By using an auxiliary pencil H'_v and calculating the cones of $V_t \cap H_{u0}$, we arrive at

(11.5) **Theorem.** *The minimal polynomial for the P-L transformation is* $P(T) = \Pi\,(T^{m_i} - 1)/(T-1)$.

(11.6) **Proof.** As in part 10, there are two cases, according as $s < n-1$ or $s = n-1$.

First, observe that (locally) V_t $(t \neq 0)$ is homeomorphic to a $(2n-2)$-ball, so in the exact sequence (10.9), ∂ is injective. Hence, the minimal polynomial of T on $d_1 = \partial D_1$ is also $P(T)$. We find d_1 by looking at the tangencies of H'_v on $V_t \cap H_{u0}$ near x.

Case 1. $s < n-1$. This is quite analogous to (10.3). $H_0 \cap B_s$ has an ordinary double point at x, so for two (small) values of v, H'_v will be tangent to $H_{u0} \cap B_s$ near x. Hence, we have two groups of $m-1$ branch points on $V_t \cap H_{u0}$ from which d_1 is formed. By induction, the minimal polynomial of T on these $2m-2$ cones is $P(T)$. Hence, the same is true for d_1 (since we know its minimal polynomial has degree $m-1$) and D_1.

Case 2. $s = n-1$. V_0 is locally $V^1 \cup \cdots \cup V^{n-1}$ and B is $(V^1 \cap V_t) \cup \cdots \cup (V^{n-1} \cap V_t)$. Hence, $V_0 \cap H_{u0}$ is $(V^1 \cap H_{u0}) \cup \cdots \cup (V^{n-1} \cap H_{u0})$ and the base of the pencil $V_t \cap H_{u0}$ is $(V^1 \cap V_t \cap H_{u0}) \cup \cdots \cup (V^{n-1} \cap V_t \cap H_{u0})$. Hence, the tangencies of H'_v on $V_t \cap H_{u0}$ near x are of two kinds:

(11.7) (1) A group of m cones with minimal polynomial $\Pi\,(T^{m_i} - 1)$ coming from the non-base-point $(V^1 \cap H_{u0}) \cap \cdots \cap (V^{n-1} \cap H_{u0})$.

(2) $n-1$ groups of $m - m_i - 1$ cones each, coming from the base-point $(V^1 \cap V_t \cap H_{u0}) \cap \cdots \cap \widehat{(V^i \cap V_t \cap H_{u0})} \cap \cdots \cap (V^{n-1} \cap V_t \cap H_{u0})$.

The minimal polynomial of T on these $(m-1)(n-1)$ cones is $\Pi\,(T^{m_i} - 1)$. Since the minimal polynomial on d_1 is of degree $m-1$, it can only be $\Pi\,(T^{m_i} - 1)/(T-1) = P(T)$ or $\Pi\,(T^{m_i} - 1)/(T+1)$. We shall eliminate the latter possibility.

First we have a result analogous to (10.10).

(11.8) **Proposition.** *Let h be a (counterclockwise) loop surrounding the $m-1$ branch points. Then h induces the transformation $T^{-(m-1)}$ if and only if the minimal polynomial is $P(T)$.*

Now to examine the action of h on the $(m-1)(n-1)$ cones of $V_t \cap H_{u0}$ we have two cases, according as the cone is induced by a base-point or not (cf. (11.7)).

V_t is given by $z_1^{m_1} \cdots z_{n-1}^{m_{n-1}} = tz_n$, L by $u = z_1 + \cdots + z_n$ and L' by $v = c_1 z_1 + \cdots + c_n z_n$ (higher powers will not affect the results). Consider first the base-point $(u, 0, \cdots, 0)$ on $V_t \cap H_u$ (as u traverses h). $z_2, \cdots, z_n$ are local coordinates there and we have

$$(11.9)\qquad \begin{aligned} V_t \cap H_u:&\quad (u - z_2 - \cdots - z_n)^{m_1} z_2^{m_2} \cdots z_{n-1}^{m_{n-1}} = tz_n, \\ L':&\quad v = c_1 u + (c_2 - c_1)z_2 + \cdots + (c_n - c_1)z_n. \end{aligned}$$

In order to put the local equation of $V_t \cap H_u$ into the standard form, let $\bar{z}_2 = z_2(u - z_2 - \cdots - z_n)^{m_1/m_2}$, so (11.9) becomes

$$(11.10)\qquad \begin{aligned} V_t \cap H_u:&\quad \bar{z}_2^{m_2} \cdots z_{n-1}^{m_{n-1}} = tz_n, \\ L':&\quad v - c_1 u = (c_2 - c_1)u^{-m_1/m_2}\bar{z}_2 + (c_3 - c_1)z_3 + \cdots + (c_n - c_1)z_n + \text{h.p.} \end{aligned}$$

By (11.3), the branch points on $V_t \cap H_u$ are given by $v - c_1 u = \text{const}\ u^{-m_1/(m-m_1-1)} t^{1/(m-m_1-1)} + \cdots$ where the constant does not depend on u. This can be written

$$v = u(c_1 + \text{const}\ u^{-(m-1)/(m-m_1-1)} t^{1/(m-m_1-1)} + \cdots).$$

We conclude that as u traverses h, the center of this group of $m - m_1 - 1$ branch points makes one circuit and its members undergo the permutation $T^{-(m-1)}$. Similarly for the other $n-2$ groups of $m - m_i - 1$ points. As for the non-base-point $(0, \cdots, 0, u)$, we eliminate z_n to get

$$\begin{aligned} V_t \cap H_u:&\quad z_1^{m_1} \cdots z_{n-1}^{m_{n-1}} = t(u - z_1 - \cdots - z_{n-1}), \\ L':&\quad v = c_n u + (c_1 - c_n)z_1 + \cdots + (c_{n-1} - c_n)z_{n-1}. \end{aligned}$$

Then from (9.12) (or directly) we find the branch points on L' to be $v = c_n u + \text{const}\,(tu)^{1/m} = u(c_n + \text{const}\ u^{-(m-1)/m} t^{1/m} + \cdots)$. Hence, this group behaves just like the other $n-1$ groups of cones, and the effect of traversing h is $T^{-(m-1)}$. Q.E.D.

12. Examples of Picard-Lefschetz transformations.

A. We give here a few examples of the P-L transformation for certain simple pencils and singularities.

The first case to consider is where the V_t are curves. The possible types of singular fibres have been classified by Kodaira [3] for genus 1 and Ogg [5] for genus 2. Kodaira worked out the P-L transformation using the modular function, but it can also be done with the previously described theory.

B. **Example 1.** $V_0 = 3$ rational lines meeting at a point. This singularity was analyzed in part 7B. We get six cones in $H_1(V_t, V_t \cap H_u)$ and $A \to B \hookrightarrow C \to A$ under T. Also $\partial A = \partial B = \partial C$ since T is the identity on $V_t \cap H_u$. Hence $A - B$ and $B - C$ form a basis for $H_1(V_t)$ and we have the matrix $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ of order 3.

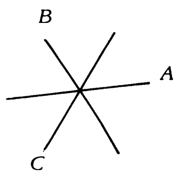

Figure 7

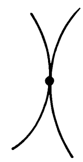

Figure 8

Example 2. $V_0 = 2$ rational lines tangent at a point. One could blow up the singularity and analyze the resulting normal crossings (viz. 11A), but it is simpler to treat it directly.

Locally, V_t is $x(x + y^2) = t$ and H_u is $u = x + y$. The Jacobian is

$$\begin{pmatrix} 2x + y^2 & 2xy \\ 1 & 1 \end{pmatrix},$$

so the branch curve is $2x + y^2 - 2xy = 0$. This is nonsingular at $(0, 0)$ and $x = -\frac{1}{2}y^2 + \cdots, u = y + \cdots, t = -\frac{1}{4}y^4 + \cdots$. Hence, the branch function is essentially $u = ct^{1/4}$, $c \neq 0$. Under T we have $A \to B \to C \to D$ and since all the boundaries are equal, $T(D) = 2D - 2C + 2B - A$. $H_1(V_t)$ is spanned by $A - B$ and $B - C$.

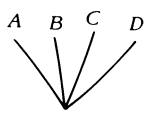

Figure 9

$T(A - B) = B - C$ and $T(B - C) = C - D$. Since $(A - B, B - C) = (B - C, C - D) = -(C - D, B - C)$, we have $T(B - C) = -(A - B)$ and the matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Example 3. $V_0 = s$ rational lines forming a closed "polygon." Each intersection is an ordinary double point, so we have s pairs of cones with

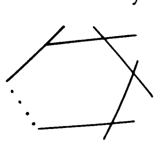

Figure 10

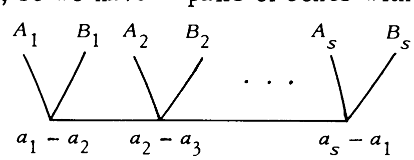

Figure 11

$\partial A_1 = \partial A_2 = a_1 - a_2$, etc. We can assume $\partial A_s = a_s - a_1$ by choosing the path correctly, because V_t is irreducible. Here, $H_1(V_t)$ is spanned by ΣA_i and

$B_1 - A_1$. Since for all i, $B_i - A_i = B_1 - A_1$ in $H_1(V_t)$, the matrix is $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

Example 4. V_0 is like Figure 12, where each component is a rational curve, four of multiplicity 1 and $s+1$ of multiplicity 2. The cones look like Figure 13 (we have ignored two sets of three). Underneath the first

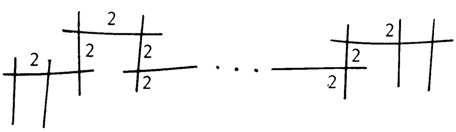

Figure 12

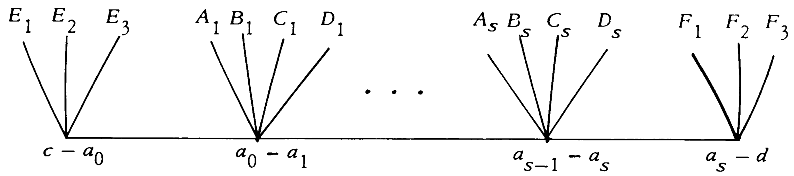

Figure 13

member of each group is its boundary, e.g., $\partial E_1 = c - a_0$, $\partial A_1 = a_0 - a_1$, etc. The boundaries of the other cones are determined by the points c, d, a_i, b_i ($i = 0, \cdots, s$) of $V_t \cap H_u$ where $T(c) = c$, $T(d) = d$, $T(a_i) = b_i$ and $T(b_i) = a_i$.

Let $G = \Sigma (A_i - B_i) + E_1 - E_2 + F_1 - F_2$ and $H = A_1 - C_1$. G and H span $H_1(V_t)$ since $(G, H) = 1$. We have $B_1 - D_1 = -A_1 - C_1$ and $E_1 - E_3 = F_1 - F_3 = 0$, so the P-L matrix is $\begin{pmatrix} -1 & s \\ 0 & -1 \end{pmatrix}$.

C. The P-L transformation for elliptic curves can be calculated more easily after expressing the general V_t as a 2-sheeted covering of the projective line, since the corresponding g_2^1 suffices to calculate $H_1(V_t)$. Of course, this is also true for any hyperelliptic pencil. This can be written in the form

(12.1) $y^2 = f(x, t)$, f a polynomial of degree $2g+1$ or $2g+2$ in x (g = genus of V_t).

The cones and their behavior can be calculated with respect to the pencil x = const. The only difficulty is that (12.1) may represent a singular surface in P^3, so must be desingularized if one wants to see what the "real" fibre V_0 is. This, of course, does not change the P-L transformation (cf. 11A).

Example 5. Consider the pencil $y^2 = x^3 + x^2 + t^s$, $s > 1$. The point (0, 0, 0) is singular, but after a finite number of quadratic transformations (in fact after $[s/2]$), the singularity is resolved and the new V_0 is just the curve of Example 3. Observe that as t moves around 0 once, t^s moves around s times. In other words the P-L transformation is the sth power of the one corresponding to the pencil $y^2 = x^3 + x^2 + t$. But V_0 here just has an ordinary double point, so the matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and its sth power is $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, as before.

Example 6. Consider Ogg's example: $y^2 = x^5 - t^s$. As s changes, the fibre V_0 on the nonsingular surface assumes various shapes, e.g., $s = 2$ and $s = 3$ are represented in Figures 14 and 15 (the components are rational lines with their corresponding multiplicities).

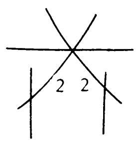

Figure 14

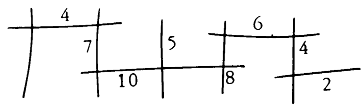

Figure 15

But as was explained above, the P-L transformation is just the sth power of the one for the pencil $y^2 = x^5 - t$. To calculate the latter we use the pencil $x = u$ and find two groups of branch points and cones, viz., $u = t^{1/5}$ and $u = \infty$. They are represented in Figure 16.

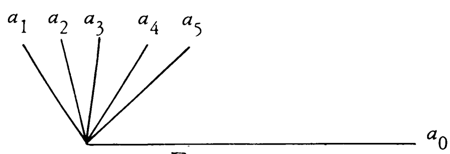

Figure 16

There is one relation in $H_1(V_t, V_t \cap H_u)$ (cf. (3.11)), viz.,

(12.2) $$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 = 0.$$

Furthermore, under T we have

(12.3) $$a_0 \to a_0, \quad a_1 \to a_2 \to a_3 \to a_4 \to a_5, \quad a_5 \to 2a_5 - 2a_4 + 2a_3 - 2a_2 + a_1.$$

If we put $A = a_1 - a_2$, $B = a_2 - a_3$, $C = a_3 - a_4$, $D = a_4 - a_5$, then A, B, C and D span $H_1(V_t)$ and the matrix is the companion matrix to the polynomial $x^4 - x^3 + x^2 - x + 1$, so is of order 10.

More generally, we have the following:

(12.4) **Proposition.** *Let $y^2 = x^m - t$. If m is odd, the matrix is companion to $(x^m + 1)/(x + 1)$, so is of order $2m$ $(m > 1)$. If m is even, the matrix is companion to $(x^m - 1)/(x^2 - 1)$, so is of order m $(m > 2)$.*

D. **Example 7.** V_t = quadrics in projective space P_n. $H_{n-1}(V_t) = 0$ if n is even, so the P-L transformation is trivial. Assume $n = 2m + 1$. Then $H_{n-1}(V_t)$ (for V_t nonsingular) is generated by two linear m-spaces A and B with the following intersection properties:

(12.5) $$\begin{aligned} &(A, A) = (B, B) = 0, \quad (A, B) = 1 \quad \text{if } m \text{ is odd;} \\ &(A, A) = (B, B) = 1, \quad (A, B) = 0 \quad \text{if } m \text{ is even.} \end{aligned}$$

By (3.7) the self-intersection number of the cycle vanishing at an ordinary double point is $2(-1)^m$. Since it must be of the form $rA + sB$ for some integers r and s, (12.5) shows that the vanishing cycle must be $A - B$. From this it follows that the P-L matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the space of dimension $n(n+3)/2$ which parametrizes the quadrics of P_n, the singular quadrics form a hypersurface of degree $n+1$. Hence, if the intersection multiplicity of the pencil V_t with this hypersurface at V_0 is even, the P-L transformation is the identity and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ otherwise.

The intersection multiplicity can be calculated as follows: choose projective coordinates so that one V_t has equation $x_0^2 + \cdots + x_n^2 = 0$. Then the multiplicity of V_0 is the number of times 0 counts as an eigenvalue of the symmetric matrix representing V_0. For instance, if the singular set of V_0 is r-dimensional and V_t is not tangent to it, the multiplicity is $r+1$.

Example 8. V_t = cubic surfaces in P_3. $H_2(V_t)$ is generated by the lines on V_t. The set of all these lines forms a surface of degree 9, since the nonsingular V_t contain just 27 lines. Hence, when t makes a loop, the 27 lines undergo some permutation, and we conclude that

(12.6) The P-L transformation for cubic surfaces in P_3 is of finite order.

For instance, as a simple special case we have

(12.7) If $V_0 = 3$ planes and V_t is otherwise general, the P-L transformation is the identity.

We can see this as follows: the pencil $V_t \cap H_u$ near $V_0 \cap H_u$ is of the type of Example 3 with $s = 3$, so there is a single invariant vanishing cycle b. Now V_0 has 3 double lines and 1 triple point. The former yield 9 double points at the base, which result in 9 invariant cones on V_t (see (11.5)). Their boundaries must be b. The triple point yields three cones: A, TA, T^2A, with boundaries a, $a+3b$, $a+6b$, where $(a, b) = -1$. The relation among these 12 cones induced by b (viz. 3D) is $(T-1)^2 A = 0$, so every cycle in V_t is invariant.

Example 9. Finally, for an example of a less trivial transformation, we mention the case of quartic surfaces in P_3, with $V_0 = 4$ planes. V_0 has 6 double lines and 4 triple points, so we get 24 invariant cones plus 4 sets of 3 on which the minimal polynomial is $(T-1)^3$.

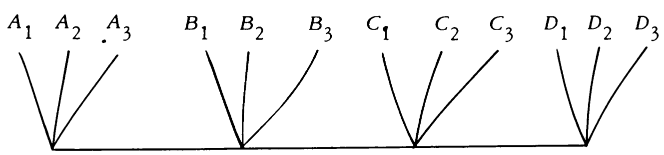

Figure 17

$(T-1)^2A_1$, $(T-1)^2B_1$, etc., are vanishing cycles which turn out to be homologous to one cycle E, while $F = A_1 + B_1 + C_1 + D_1$ is a cycle such that $(E, F) = -1$. Hence, $(T-1)^2F = 4E \neq 0$, and we conclude

(12.8) The minimal polynomial of the P-L transformation on $H_2(V_t)$ is $(T-1)^3$.

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APPENDIX AND SUPPLEMENTARY BIBLIOGRAPHY (PHILLIP A. GRIFFITHS)

1. **Restatement of the mondromy theorem.** Let $D = \{t \in \mathbf{C}: |t| < 1\}$ be the unit disc in $\mathbf{C}$, X a complex manifold of dimension $n+1$ which admits a projective embedding, and $f: X \to D$ a proper, holomorphic mapping which has $t = 0$ as the only critical value. Then $V_t = f^{-1}(t)$ is a smooth, projective variety for $t \neq 0$, and by Hironaka we may assume that V_0 has normal crossings. This is the localization in the parameter space of the situation $f: W \to C$ of the above paper. We write

$$V_0 = m_1 D_1 + \cdots + m_l D_l$$

where the D_j are smooth divisors meeting transversely. We then define

$$F_s(V_0) = \sum_{i_1 < \cdots < i_s} D_{i_1} \cap \cdots \cap D_{i_s},$$

so that the following hold:

$$F_1(V_0) = |V_0| \text{ is the support of } V_0;$$

$$F_{s+1}(V_0) \subset F_s(V_0); \quad \text{and}$$

$$F_s(V_0) = \text{points of order } s \text{ on } |V_0|.$$

The two numerical characters

$$\text{(A.1)} \qquad \begin{cases} N = \text{l.c.m.}\,(m_1, \cdots, m_l), \\ r_0 = \max\limits_s \{F_s(V_0) \neq \emptyset\} \leq n+1 \end{cases}$$

are associated to the situation $f\colon X \longrightarrow D$.

Fix $t_0 \neq 0$ and set $V = V_{t_0}$, $H^q(V) = H^q(V, \mathbf{Q})$, and denote by $T\colon H^q(V) \longrightarrow H^q(V)$ the Picard-Lefschetz transformation given by the generator of $\pi_1(D - \{0\})$ acting on the cohomology $H^q(V)$. Theorems I$'$, II$'$ of the paper may be summarized by the matrix equation

$$\text{(A.2)} \qquad (T^N - I)^{r_1} = 0 \qquad (r_1 = \min(r_0, q+1)).$$

This result is generally referred to as the *monodromy theorem*. Since Landman's thesis there have been several proofs of (A.2) of widely varying natures, and our purpose is to present a short bibliography to these proofs.

2. **Geometric proofs.** These have been given by Clemens [11] and Deligne-Grothendieck [13]. The idea is to consider the situation

$$\begin{array}{ccc} X & \xrightarrow{g} & V_0 \\ \downarrow f & & \downarrow f \\ D & \longrightarrow & \{0\} \end{array}$$

where g is the retraction of X onto the singular fibre V_0. Thus $g\colon V_t \longrightarrow V_0$ may be viewed as a sort of "collapsing" map. By using the normal crossings of V_0, we may assume that $f \circ g = f$. If we now apply the Leray spectral sequence to $X \xrightarrow{g} V_0$, then f operates in this via its operation on the Leray direct image sheaves $R^q_{g_*}(\mathbf{Q})$. The monodromy theorem follows by examining this action. A further discussion of this is given in § 15 of [14].

There is also a local version of the monodromy theorem when the fibres V_t are hypersurfaces in $\mathbf{C}^{n+1}$ acquiring an isolated singular point. Early results here by Pham, Brieskorn, and Milnor led to a fascinating interplay between the P-L transformation and exotic spheres [16].

3. **Proofs using the Picard-Fuchs equation.** The sheaf $\mathcal{H}^q = R^q_{f_*}(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_D$ admits a canonical flat connection ∇, the *Gauss-Manin connection*, such that

$R^q_{f_*}(\mathbf{C})$ forms the subsheaf of flat sections. Now $\mathcal{H}^q$ extends to a coherent sheaf on D, but ∇ will generally have a singularity at $t = 0$ whose "residue" is $\log T$. The detailed study of $(\mathcal{H}^q, \nabla)$ led Brieskorn [10], Deligne [12], and Katz [15] to a proof of the monodromy theorem. Moreover, Katz was able to refine the *index of unipotency* r in (A.2) as follows: Using the *Hodge decomposition* [18],

$$\text{(A.3)} \qquad \begin{cases} H^q(V, \mathbf{C}) = H^{q,0}(V) \oplus H^{q-1,1}(V) \oplus \cdots \oplus H^{0,q}(V), \\ \qquad = H^{q-a,a}(V) = \overline{H^{a,q-a}(V)}, \end{cases}$$

we define the integer r_2 by

$$r_2 = \min(r_0, s) \quad \text{where } H^{a,q-a}(V) = 0 \quad \text{for } a > s.$$

Then $r_0 \leq q + 1$, and this number measures how many terms vanish from the "outside" of the Hodge decomposition (A.3). Katz's refinement of the monodromy theorem is that

$$\text{(A.4)} \qquad (T^N - I)^{r_2} = 0.$$

This improvement is of an analytic rather than a geometric nature.

4. **Analytic proofs.** Associated to the situation $f\colon X \longrightarrow D$ there is a *classifying space for Hodge structures* (= period matrix domain) $\mathfrak{D} = G/H$ such that $T \in G$ and such that there is a holomorphic mapping

$$\text{(A.5)} \qquad D \xrightarrow{\ \phi\ } \mathfrak{D}/\{T^k\}$$

where $\phi(t) = \{\text{Hodge decomposition of } H^q(V_t, \mathbf{C})\}$ [14]. Passing to the universal covering $\mathcal{H} = \{z = x + \sqrt{-1}\, y,\ y > 0\}$ of D, the *period mapping* (A.5) lifts to give

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \Phi\ } & \mathfrak{D} \\ \downarrow & & \downarrow \\ D & \xrightarrow{\ \phi\ } & \mathfrak{D}/\{T^k\} \end{array}$$

where $\Phi(z + 1) = T \cdot \Phi(z)$. The mapping Φ is distance decreasing relative to the Poincaré metric on $\mathcal{H}$ and a suitable G-invariant metric on $\mathfrak{D}$. Using this, Borel (unpublished) gave a very simple proof that the eigenvalues of T are roots of unity. Recently, this approach has been greatly extended by Schmid [17] who has given a complete asymptotic analysis of the Hodge decomposition $\phi(t)$ as $|t| \longrightarrow 0$. As a by-product, Schmid obtains the exact position of $\log T$ in the Lie algebra of G giving yet another proof of the strong monodromy theorem (A.4). In addition to this arithmetic property, Schmid finds that T has very remarkable positivity properties, the simplest being that given by (13.5) on p. 266 of [14].

5. **Computational questions.** The problem of computing the P-L transformation in explicit examples is generally quite difficult. It seems to me that the global algorithmic procedure given by Landman in this paper is by far the best general method, as is perhaps suggested by the fairly complicated examples he is able to treat with relative ease. In particular, all of the P-L transformations which I have seen arising from the study of Feynman integrals in mathematical physics should be accessible to these techniques.

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